IB Mathematics: Analysis and Approaches

1.1 Number Toolkit

Scientific Notation & Laws of Indices

Maths With Lemon
squeezing the knowledge

DP Analysis & Approaches (SL/HL)

Mathematician:______________________________

Contents

1 Content Guidance

This lesson focuses on operations with numbers in the form

$$a \times 10^k$$

where

$$1 \leq a < 10$$

and

$$k \in \mathbb{Z}.$$

Calculator or computer notation is not acceptable in IB examinations. For example,

$$5.2E30$$

is not acceptable and should be written as

$$5.2 \times 10^{30}.$$

2 Warm-Up: Exponents Challenge

1) Express each as a single power of 3.

   a) $(3^5)^3$
   b) $3^5 \cdot 3^3$
   c) $3^5 \cdot 3$
   d) $\dfrac{3^5}{3^3}$
   e) $\dfrac{3^5}{3}$
   f) $3^{1/3} \cdot 3^{1/5}$
   g) 9^2
   h) 27^3
   i) $\left(\dfrac{1}{3}\right)^3$

2) Solve:

$$4^{2x-7} = 16$$

3) Write the following without brackets or negative exponents:

$$(2ab^{-2})^{-3}$$

3 Scientific Notation

3.1 Definition

Scientific notation is a method of writing very large or very small numbers using powers of ten.

A number is written in scientific notation if it has the form

$$a \times 10^k$$

where

$$1 \leq a < 10$$

and

$$k \in \mathbb{Z}.$$

It is also known as:

- standard form,
- standard index form,
- scientific notation.

3.2 Why Use Scientific Notation?

Scientific notation allows us to:

- write very large and very small numbers neatly,
- compare numbers more easily,
- perform calculations efficiently,
- display numbers that may not fit on a calculator screen.

3.3 Basic Examples

$$756000 = 7.56 \times 10^5$$

$$0.000021 = 2.1 \times 10^{-5}$$

$$4500000000 = 4.5 \times 10^9$$

$$0.00000034 = 3.4 \times 10^{-7}$$

4 Converting Numbers

Example 1

Convert each number to scientific notation.

a) 276000

b) 0.000015

c) 260000

d) 0.00043

Solutions

a) $276000 = 2.76 \times 10^5$

b) $0.000015 = 1.5 \times 10^{-5}$

c) $260000 = 2.6 \times 10^5$

d) $0.00043 = 4.3 \times 10^{-4}$

Example 2

Convert each scientific number into ordinary form.

a) 7.12×10^4

b) 2.54×10^{-5}

Solutions

a) $7.12 \times 10^4 = 71200$

b) $2.54 \times 10^{-5} = 0.0000254$

5 Calculations with Scientific Notation

5.1 Multiplication

When multiplying numbers in scientific notation, multiply the coefficients and add the powers of ten:

$$(a \times 10^m)(b \times 10^n) = ab \times 10^{m+n}.$$

Example 3

Without a calculator, write

$$(3 \times 10^7)(4 \times 10^{-3})$$

in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}$.

Solution

$$(3 \times 10^7)(4 \times 10^{-3}) = 12 \times 10^{7+(-3)}$$

$$= 12 \times 10^4$$

This is not yet in scientific notation because 12 is not between 1 and 10.

$$12 \times 10^4 = 1.2 \times 10^5$$

$$\boxed{1.2 \times 10^5}$$

5.2 Division

When dividing numbers in scientific notation, divide the coefficients and subtract the powers of ten:

$$\frac{a \times 10^m}{b \times 10^n} = \frac{a}{b} \times 10^{m-n}.$$

Example 4

Show that if

$$x = 3 \times 10^7$$

and

$$y = 4 \times 10^{-2},$$

then

$$\frac{x}{y} = 7.5 \times 10^8.$$

Solution

$$\frac{x}{y} = \frac{3 \times 10^7}{4 \times 10^{-2}}$$

$$= \frac{3}{4} \times 10^{7-(-2)}$$

$$= 0.75 \times 10^9$$

This is not yet in scientific notation because $0.75 < 1$.

$$0.75 \times 10^9 = 7.5 \times 10^8$$

$$\boxed{7.5 \times 10^8}$$

5.3 Addition and Subtraction

To add or subtract numbers in scientific notation, first make the powers of ten the same.

Example 5

Show that

$$3.2 \times 10^{19} + 4.5 \times 10^{20} = 4.82 \times 10^{20}.$$

Solution

Rewrite the first number using 10^{20}:

$$3.2 \times 10^{19} = 0.32 \times 10^{20}.$$

Therefore,

$$\begin{aligned} 3.2 \times 10^{19} + 4.5 \times 10^{20} &= 0.32 \times 10^{20} + 4.5 \times 10^{20} \\ &= (0.32 + 4.5) \times 10^{20} \\ &= 4.82 \times 10^{20}. \end{aligned}$$

$$\boxed{4.82 \times 10^{20}}$$

6 Applications of Scientific Notation

Example 6

The area of Africa is approximately

$$3.04 \times 10^{13} \text{ m}^2.$$

The area of Europe is approximately

$$1.02 \times 10^{13} \text{ m}^2.$$

The population of Africa is approximately 1.2 billion and the population of Europe is 741 million.

a) How many times bigger is Africa than Europe?

b) Write the population of Europe in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}$.

c) Does Africa or Europe have more people per square metre? Justify your answer.

Solution

a)

$$\frac{3.04 \times 10^{13}}{1.02 \times 10^{13}} = \frac{3.04}{1.02} \approx 2.98$$

Africa is approximately $\boxed{2.98}$ times bigger than Europe.

b)

$$741 \text{ million} = 741000000$$

$$741000000 = 7.41 \times 10^8$$

$$\boxed{7.41 \times 10^8}$$

c) Africa's population density is

$$\frac{1.2 \times 10^9}{3.04 \times 10^{13}} \approx 3.95 \times 10^{-5}.$$

Europe's population density is

$$\frac{7.41 \times 10^8}{1.02 \times 10^{13}} \approx 7.26 \times 10^{-5}.$$

Since

$$7.26 \times 10^{-5} > 3.95 \times 10^{-5},$$

Europe has more people per square metre.

$$\boxed{\text{Europe has the higher population density.}}$$

7 Generalising Scientific Notation

Example 7

You are given that

$$(3 \times 10^a)(5 \times 10^b) = c \times 10^d$$

where $1 \leq c < 10$ and $d \in \mathbb{Z}$.

a) Find the value of c.

b) Find an expression for d in terms of a and b.

Solution

$$(3 \times 10^a)(5 \times 10^b) = 15 \times 10^{a+b}$$

Since 15 is not between 1 and 10,

$$15 \times 10^{a+b} = 1.5 \times 10^{a+b+1}.$$

Therefore,

$$\boxed{c = 1.5}$$

and

$$\boxed{d = a + b + 1}.$$

Example 8

You are given that

$$(2 \times 10^a) \div (5 \times 10^b) = c \times 10^d$$

where $1 \leq c < 10$ and $d \in \mathbb{Z}$.

a) Find the value of c.

b) Find an expression for d in terms of a and b.

Solution

$$(2 \times 10^a) \div (5 \times 10^b) = \frac{2}{5} \times 10^{a-b}$$

$$= 0.4 \times 10^{a-b}.$$

Since $0.4 < 1$, rewrite it as

$$0.4 \times 10^{a-b} = 4 \times 10^{a-b-1}.$$

Therefore,

$$\boxed{c = 4}$$

and

$$\boxed{d = a - b - 1}.$$

Example 9

Let

$$x = a \times 10^p$$

and

$$y = b \times 10^q$$

where

$$4 < a < b < 9.$$

When xy is written in scientific notation,

$$xy = c \times 10^r.$$

Express r in terms of p and q.

Solution

$$xy = (a \times 10^p)(b \times 10^q) = ab \times 10^{p+q}.$$

Since

$$4 < a < b < 9,$$

then

$$16 < ab < 81.$$

So ab is between 10 and 100. Therefore it must be rewritten as a number between 1 and 10 multiplied by one extra power of 10.

Hence,

$$r = p + q + 1.$$

$$\boxed{r = p + q + 1}$$

8 Theory of Knowledge

Scientific notation changes the way a number is represented, but not the value it represents.

A core principle of mathematics is the relationship between:

- representation,
- modelling,
- equivalence.

These ideas are all relevant when expressing numbers using scientific notation.

Knowledge Question

> How much can you change a piece of original knowledge before it becomes new knowledge?

When you change the way a number is represented, is it still the same number, or does it become something new?

This connects to the paradox of Theseus' Ship. If every part of a ship is replaced over time, is it still the same ship?

In mathematics, changing the representation of an object does not necessarily change the object itself.

9 Laws of Indices

Laws of indices allow us to simplify expressions involving powers.

An exponent is a power to which a number or expression is raised. The number being raised is called the base.

The index laws are especially useful when working with scientific notation, logarithms, functions, and algebraic expressions.

9.1 Index Laws Summary

The following laws apply when the bases are the same.

$$x^m \times x^n = x^{m+n}$$

$$x^m \div x^n = x^{m-n}$$

$$(x^m)^n = x^{mn}$$

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$x^1 = x$$

$$x^0 = 1$$

$$x^{-n} = \frac{1}{x^n}$$

$$x^{1/n} = \sqrt[n]{x}$$

$$x^{m/n} = \sqrt[n]{x^m}$$

9.2 Important Note

Index laws only work directly when the bases are the same.

For example,

$$9^4 = (3^2)^4 = 3^8.$$

Therefore,

$$9^4 \div 3^7 = 3^8 \div 3^7 = 3.$$

10 Worked Examples: Laws of Indices

Example 10

Simplify:

$$\frac{(3x^2)(2x^3y^2)}{6x^2y}.$$

Solution

$$(3x^2)(2x^3y^2) = 6x^5y^2.$$

So,

$$\frac{(3x^2)(2x^3y^2)}{6x^2y} = \frac{6x^5y^2}{6x^2y}.$$

Cancel common factors:

$$= x^{5-2}y^{2-1} = x^3y.$$

$$\boxed{x^3y}$$

Example 11

Simplify:

$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}.$$

Solution

$$(4x^2y^{-4})^3 = 4^3x^6y^{-12} = 64x^6y^{-12}.$$

Also,

$$(2x^3y^{-1})^{-2} = 2^{-2}x^{-6}y^2 = \frac{1}{4}x^{-6}y^2.$$

Now multiply:

$$64x^6y^{-12} \cdot \frac{1}{4}x^{-6}y^2$$

$$= 16x^{6-6}y^{-12+2} = 16x^0y^{-10}.$$

Since $x^0 = 1$,

$$= 16y^{-10}.$$

Rewrite using positive indices:

$$\boxed{\frac{16}{y^{10}}}$$

11 Mixed Practice Questions

1) Write 45000000 in scientific notation.
2) Write 0.00000072 in scientific notation.
3) Convert 6.21×10^5 into ordinary form.
4) Convert 8.04×10^{-4} into ordinary form.
5) Calculate:

$$(2.4 \times 10^6)(3 \times 10^{-2})$$

6) Calculate:

$$\frac{6 \times 10^8}{2 \times 10^3}$$

7) Calculate:

$$4.7 \times 10^9 + 2.1 \times 10^8$$

8) Simplify:

$$x^4 \cdot x^7$$

9) Simplify:

$$\frac{a^9}{a^3}$$

10) Simplify:

$$(2x^3y^{-2})^2$$

11) Rewrite without negative indices:

$$5x^{-3}y^2$$

12) Simplify:

$$\left(\frac{3a^2}{b^{-1}}\right)^2$$

12 Exam Tips

- Scientific notation answers must have exactly one non-zero digit before the decimal point.
- Calculator notation such as `E` notation must be rewritten using powers of ten.
- To add or subtract numbers in scientific notation, first match the powers of ten.
- To multiply numbers in scientific notation, multiply the coefficients and add the powers.
- To divide numbers in scientific notation, divide the coefficients and subtract the powers.
- Index laws only work directly when the bases are the same.
- Negative powers should usually be rewritten as fractions in final answers.
- Index laws are often needed in logarithms, functions, polynomials, and calculus.